Problem 5

Compute

$$\frac{1}{1} + \frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+\cdots+100}.$$

Solution. Note that $1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ for any natural number k. Therefore, we may write

$$\frac{1}{1+\cdots+k} = \frac{2}{k(k+1)} = 2\left(\frac{1}{k+1} - \frac{1}{k}\right).$$

Thus

$$
\begin{aligned}
\frac{1}{1} + \frac{1}{1+2} + \cdots + \frac{1}{1+\cdots+100} &= \sum_{k=1}^{100} 2\left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= 2\left((\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{99} - \frac{1}{100}) + (\frac{1}{100} - \frac{1}{101})\right) \\
&= 2\left(1 - \frac{1}{101}\right) \\
&= \frac{200}{101}.
\end{aligned}
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$\blacksquare$